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A Kinetic Theory, Near-Continuum Model for Membrane Transport

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Continuum flow theory (Poiseuille flow, Fick diffusion) is of dubious validity for small pores. For gases, kinetic theory permits an extension of continuum theory to Knudsen near-continuum slip-flow, by means of a change only in boundary conditions. The range of validity of near-continuum gas flow theory is for channel widths larger than a small number of mean free paths. Here a comparable kinetic theory extension of continuum theory has been developed for liquids. A "saltation," near rolling, boundary condition is shown to be applicable. The range of validity of this near continuum liquid flow theory is for channels wider than about five molecular diameters (i.e., 15 Å for water). Predictions of the theory are tested against the well calibrated data of Beck and Schultz (1972) as well as data from biological membranes. The results are compared with the continuum theory used by Renkin (1954) and others.

INTRODUCTION

The problem undertaken in this paper is the description of the flow of water and of various sized solute molecules through natural and artificial membranes. For small membrane pores or channels, Navier–Stokes viscous flow theory and Fick diffusion theory are of dubious validity, whereas they are correct above some critical size. We propose to apply the results of a kinetic theory of liquids, developed at length elsewhere (Iberall and Schindler, 1973) to membrane pores and to reconcile flow theories for large and small pores. Finally, we compare those results with experimental work.

In 1951, Pappenheimer et al. (1951) applied continuum hydrodynamic results (Fick diffusion, Poiseuille flow) to compute pore sizes in natural membranes. Later, Renkin (1954) suggested that the diffusion through membranes D_m, relative to that in pure water D_0, can be described by a combination of two factors. The first is the steric hindrance proposed by Ferry (1936) and represents the fraction of the pore cross section area available to the diffusing molecule. It is assumed that any molecule which hits the lip of the pore cannot enter the pore. The second factor was derived by Faxen (1922) for a large molecule moving through fluid-filled channels or tubes. It represents the effect of the channel walls in changing the Stokes drag coefficient used to describe such motion in the absence of nearby walls. Renkin combined these results and obtained

$$\frac{D_m}{D_0} = \left[1 - \frac{R_g}{R_p}\right]^2 \left[1 - 2.10\,\frac{R_g}{R_p} + 2.09\,\frac{R_g^{\,3}}{R_p^{\,3}} - 0.95\,\frac{R_g^{\,5}}{R_p^{\,5}}\right], \tag{1}$$

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where R_s = radius of solute molecule, R_p = radius of (assumed circular) pore.

Renkin's equation has been widely used in studies of solute flows through membranes. However, it is well known that for small pores the boundary conditions assumed in the derivation of Eq. (1) are not valid and, in fact, the entire Navier–Stokes analysis may not be valid. Below some minimal pore size, one must modify either boundary conditions or equations sets or both. However, that minimal pore size is still a matter of discussion (Beck and Schultz, 1972; Levitt, 1973). In the appendix we suggest that the Navier–Stokes equation set, with the appropriate boundary conditions, may be valid to pores as small as 15 Å.

Can a strictly continuum theory for this phenomenon be developed? In a recent paper Mikulecky (1972) attempted to apply a continuum hydrodynamics formalism and to model the wall-fluid interaction. He assumed "zero" velocity and velocity gradient at the walls. While he is able to get an analytic result for the velocity profile, he states that "the continuum mechanical approach . . . like the phenomenological theory of non-equilibrium thermodynamics . . . leads to no direct interpretation of mechanisms" and that "the presence of velocity gradients presents a severe limitation to the application of most popular versions of nonequilibrium thermodynamic formalism." That formalism, we have shown, is equivalent to the Navier–Stokes equations. Thus, one is compelled to develop a kinetic theory for flow processes in narrow spaces.

For gas flows, the modifications required for small pore sizes are well understood. As pore sizes grow smaller, one proceeds from Poiseuille's law to the Knudsen slip-flow law (via a change in boundary conditions) to free molecular flow (via an entirely different theory). In the last case, continuum hydrodynamics no longer applies at all, and one must resort to a purely kinetic theory. For the Knudsen slip-flow, in the near-continuum regime, a combination of flow equations from continuum theory and boundary conditions from kinetic theory provides an adequate model (Kennard, 1939). In this paper, we seek the liquid theory counterpart to Knudsen gas slip-flow. Thus, the Navier–Stokes equations are assumed to be valid (see Appendix), while new boundary conditions, based on kinetic theory, are sought.

BOUNDARY CONDITIONS

The following view, based on a kinetic theory (Iberall and Schindler, 1973), can be used to obtain suitable boundary conditions between the flowing water (and/or solute) and the pore walls. The key point is that water molecules which strike a "solid" surface generally tend to stick to it and "smooth" it out. Then the next layer of water molecules rolls over the smoothed surface. (The sides of the pore are generally rough. Thus, we view this first layer as coating and smoothing the pore surface, thereby producing an effective smoothed pore radius or slit width which we identify as the measurable quantity.)

When this boundary condition is applied to a pore wide enough that the Navier–Stokes equations hold, the velocity profile across the pore is

$$v(r) = \frac{g}{4\mu}\,(R_p{}^2 - r^2) + v_{\text{wall}}, \tag{2}$$

where $r =$ distance from center of pore, $v_{\text{wall}} =$ velocity of rolling layer near the wall, $g =$ pressure gradient (dynes/cm^2/cm), $\mu =$ viscosity (gm/cm/sec). When a molecule moves from the layer next to the rolling layer into the rolling layer, its velocity tangent to the surface changes by

$$\Delta v = v(R_p - 2R_w) - v_{\text{wall}} = \frac{gR_p^2}{\mu}\left[\frac{R_w}{R_p} - \left(\frac{R_w}{R_p}\right)^2\right],$$

where $R_w =$ radius of a water molecule. If

$$\Delta v = fv(R_p - 2R_w),$$

where $f =$ fraction of moment lost, then

$$\left(\frac{f}{1-f}\right) v_{\text{wall}} = \Delta v.$$

Then

$$v_{\text{wall}} = \left(\frac{1-f}{f}\right)\frac{gR_p^2}{\mu}\left[\frac{R_w}{R_p} - \left(\frac{R_w}{R_p}\right)^2\right].$$

The mean volume flow across the section is

$$Q = \int_0^{R_p} 2\pi r v(r)\,dr$$

or

$$Q = \frac{\pi R_p^4 g}{8\mu}\left\{1 + 8\left(\frac{1-f}{f}\right)\left[\frac{R_w}{R_p} - \left(\frac{R_w}{R_p}\right)^2\right]\right\}. \tag{3}$$

Estimation of the fractional transfer of momentum is a very difficult theoretical problem. It is complicated by lack of knowledge of the intermolecular forces between fluid and wall molecules, the shapes of the molecules, the coefficients of rolling or sliding friction of one molecule past a layer of molecules. To provide a start in the problem, here we present a very elementary model which, hopefully, is not too far removed from some reasonable effect of the various physical processes involved. Thus, we make a simple application of the laws of conservation of energy and momentum. We assume that in moving into the rolling layer, a molecule loses linear momentum and gains angular momentum under the influence of an assumed impulse J (Fig. 1):

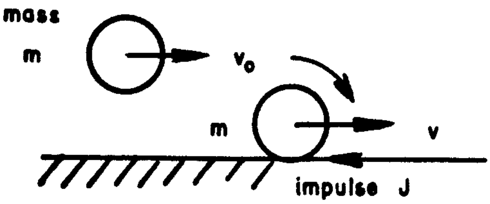

FIG. 1. The "saltation" of molecules at the pore walls.

$$\textit{linear momentum:}\quad mv = mv_0 - J,$$
$$\textit{angular momentum:}\quad (2\, mR_w^2/5)\ (v/R_w) = JR_w,$$

where (v/R_w) = angular velocity, $(2\, mR_w^2/5)$ = moment of inertia of a sphere. Then, very easily

$$v = 5\ v_0/7 \tag{4}$$

so that

$$f = (v_0 - v)/v_0 = 2/7.$$

(Note: in a simpler "kinematic" nonsaltation model, one would have presumed $f = 1/2$ whereby the rolling molecule can accommodate to both the wall and its neighboring "sliding" layer. This points up the need for more work on the details of this calculation. The factor $(1 - f)/f$ is rather sensitive to the value of f.)

Energy is not conserved; it is lost to the frictional impulsive force J.

This value of $f = 2/7$ can then be applied to the flow equation (3) yielding

$$Q = \frac{\pi R_p^4 g}{8\mu}\left\{1 + 20\frac{R_w}{R_p} - 20\frac{R_w^2}{R_p^2}\right\}.$$

Finally, when the geometric hindrance factor is added (Renkin, 1954)

$$Q = \frac{\pi R_p^4 g}{8\mu}\left\{1 + 20\frac{R_w}{R_p} - 20\frac{R_w^2}{R_p^2}\right\}\left\{1 - \frac{R_w}{R_p}\right\}^2. \tag{5}$$

(The factor for drag correction, $(1 - 2.10\ R_w/R_p + \cdots)$, is now omitted because the effect of nonzero velocities at the walls is already taken into account, and more than a simple modification is required to account only for the boundary conditions at the surface of each water molecule.)

APPLICATION

The flow equation (5) is proposed to be valid for both artificial and natural membranes for which the Navier–Stokes equations with modified (here, rolling) boundary conditions provide an adequate description. Equation (5) can be tested with the data found in the experimental paper of Beck and Schultz (1972). Because of their control of the geometric pore configuration, the test of the theory will be much sharper than for natural membranes. For their study they (a) made their own microporous membranes using fission fragments from U^{235}; (b) measured pore sizes by water flow, air flow, and electron microscopy; and (c) measured the diffusion of various solutes through these pores. The water flow measurement of pore radius was made using Poiseuille's law, with zero velocity boundary conditions at the walls

$$Q = \frac{\pi g R_{p_0}^4}{8\mu}, \tag{6}$$

where R_{p_0} = computed pore radius.

Our modified flow formula, Eq. (5), can be used to obtain corrections to the pore

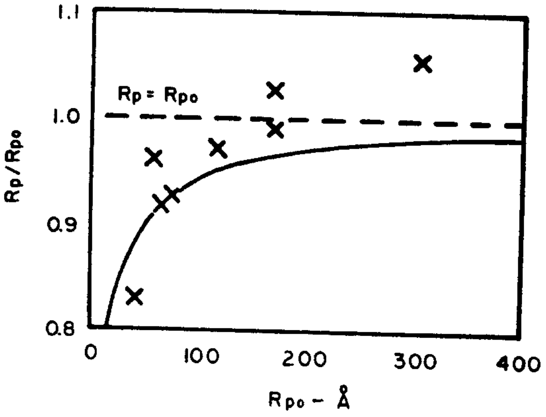

FIG. 2. Ratio of a corrected to the Poiseuillean pore radius R_p/R_{p_0} vs Poiseuillean pore radius R_{p_0} for water flow. Solid line is theoretical result (Eq. (7)). Crosses are the ratios of experimental radii, from air flow measurements and Knudsen slip-flow, to Poiseuillean pore radii R_p/R_{p_0} vs R_{p_0} (from measurements of Beck and Schultz (1972)).

radii R_{p_0} obtained from this simple equation. Since the flows are equal

$$R_{p_0}^4 = R_p^4 \left\{1 + 20\frac{R_w}{R_p} - 20\frac{R_w^2}{R_p^2}\right\}\left\{1 - \frac{R_w}{R_p}\right\}^2. \qquad (7)$$

The reliability of this relation can be checked with the air flow data of Beck and Schultz (1972). Pore radii were determined from the well established Knudsen gas slip-flow result. Generally, the radii derived from the air flow measurements were smaller than those from the water measurements. Figure 2 shows their R_p (air)/R_{p_0} (water) vs R_{p_0} (water). Also shown is R_p/R_{p_0} vs R_{p_0} from Eq. (7). The theoretically

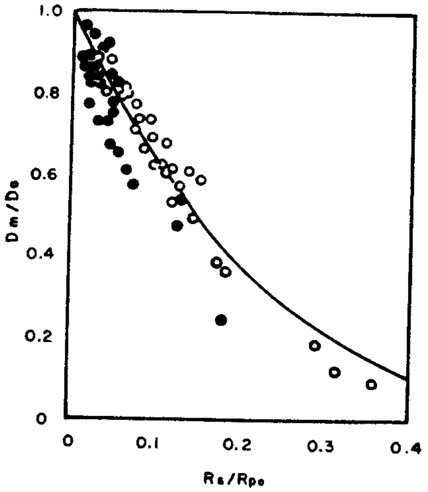

FIG. 3. Data of Beck and Schultz ((1972), circles) compared to Renkin theory (solid line) for the ratio of apparent to bulk diffusion coefficients D_m/D_0, for small pores ($R_p < 100$ Å, open circles) and large pores ($R_p > 100$ Å, filled circles).

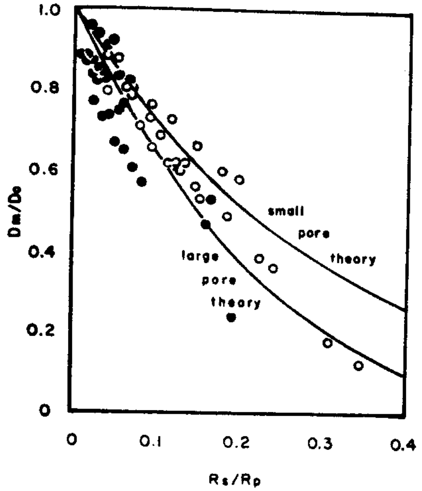

FIG. 4. Data of Beck and Schultz (1972), corrected by theory (Eq. (7)) for small pores ($R_p < 100$ Å, open circles) and large pores ($R_p > 100$ Å, filled circles), compared to Renkin large pore theory and proposed small pore theory (Eq. (8)) for D_m/D_0.

corrected pore radii are in fair agreement with the radii obtained from the air flow measurements, especially for the smaller pores where we expect the new theory to be better than the simple Poiseuille law.

Beck and Schultz measured the ratio D_m/D_0 for several solute and pore sizes and compared the results to the Renkin formula, Eq. (1). Their data are given in Fig. 3, where D_m/D_0 is plotted against R_s/R_{p_0}. When the correction above is applied to R_s/R_{p_0}.

$$\left(\frac{R_s}{R_{p_0}}\right)^4 = \left(\frac{R_s}{R_p}\right)^4 \left\{1 + 20\,\frac{R_s}{R_p} - 20\,\frac{R_s{}^2}{R_p{}^2}\right\}^{-1} \left\{1 - \frac{R_s}{R_p}\right\}^{-2}$$

a new graph of D_m/D_0 vs R_s/R_p can be made from the Beck and Schultz data (Fig. 4).

The corrected data for the larger pores (filled circles) now fall somewhat closer to the Renkin curve. This is not surprising because that curve should be valid for large pores, where the Navier–Stokes equations, with zero velocity boundary conditions, are certainly valid.

Data for small pore (open circles) sizes (or appreciable R_s/R_p) do not fit Renkin's curve so well. Here the boundary conditions applicable to the diffusing solute molecules should be taken into account.

SMALL PORE THEORY

For small pores, the theory for the flow of large solute molecules must be re-examined. The small water molecules maintain an essentially Navier–Stokes flow (with proper boundary conditions) perhaps down to $R_p \approx 15$ Å (see Appendix).

However, each solute molecule proceeds by means of a "random walk" and, because of the narrowness of the pore, makes many interactions with the pore walls during its passage. We will assume the velocity profile *of the solute molecules* to be the following: In the "core" of the pore the molecules have an essentially constant diffusive velocity v_0, while at the wall they roll with a velocity v_{wall}. By analogy with the water flow boundary condition, we assume that v_{wall} and v_0 are related by Eq. (4), namely, $v_{\text{wall}} = 5/7\ v_0$. The mean *solute flow* velocity $\bar{v}$ is the area weighted average

$$\bar{v} = \frac{v_0 \pi (R_p - 2R_s)^2 + 5\ v_0 \pi [R_p{}^2 - (R_p - 2R_s)^2]/7}{\pi R_p{}^2}$$

so

$$\frac{\bar{v}}{v_0} = 1 - \frac{8}{7}\frac{R_s}{R_p} + \frac{8}{7}\frac{R_s{}^2}{R_p{}^2}.$$

Finally, noting that D_m/D_0 is proportional to $\bar{v}/v_0$ and adding the steric hindrance factor give

$$\frac{D_m}{D_0} = \left\{1 - \frac{8}{7}\frac{R_s}{R_p} + \frac{8}{7}\frac{R_s{}^2}{R_p{}^2}\right\}\left\{1 - \frac{R_s}{R_p}\right\}^2. \tag{8}$$

This curve is shown in Fig. 4. Most of the Beck and Schultz data lie between it and the Renkin curve.

SUMMARY

The two curves in Fig. 4 represent limiting theories for solute flow through microporous membranes. The Renkin curve is valid when R_s/R_p is small enough that wall velocities are essentially negligible. The other curve is derived by the introduction of nonzero velocity boundary conditions (here specifically, rolling) at the walls of the pores.

The same theories are presumed to be applicable to permeability studies of natural membranes. We have collected a fair sampling of natural membrane data for comparison with this theory. However, the large scatter in the experimental data inhibits us from presenting those results here. (In addition, a second issue is involved: whether in living membranes one is dealing with pore, slits, or some other structure of the imagination.) The natural membrane data would tend only to obscure our main conclusion, namely, that a consistent theory for flow in large and small pores can be developed, partly out of continuum theory, and partly out of a kinetic theory.

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APPENDIX

The Navier–Stokes equations are based on a continuum hydrodynamic formalism. Their validity, therefore, depends on there being sufficient space and time available that the spatial and temporal averages required for the continuum description can be adequately performed. Gases and liquids present different requirements for this averaging, the gas case being considerably simpler.

Gases. Gases are characterized by a mean free path l and a collision time τ. As long as a space larger than about $10l$ and a time greater than about 10τ are provided, the Navier–Stokes equations are valid. For the typical ideal gas, $10l \approx 5000$ Å and $10\tau \approx 10^{-9}$. Between $10l$ and $3l$ and/or 10τ and 3τ, continuum (or, now, near-continuum) hydrodynamics can still be used, but with modified boundary conditions (e.g., slip-flow).

Liquids. For liquids, the temporal requirements become more complicated. Instead of only one time scale, there are now three (Iberall and Schindler, 1973): the time that it takes a molecule to vibrate in the intermolecular potential well of its surrounding neighbors (depending on intermolecular potential, molecular size and mass, and number of near neighbors), $\tau_1 = 2 \times 10^{-13}$ sec (for water); the time that it takes to escape from that "cage" (depending on τ_1 and fluctuation probabilities) $\tau_2 = 4 \times 10^{-12}$ sec; the time that it takes to diffuse one molecular diameter, the analogous "mean free path" (depending on molecular size and diffusion constant) $\tau_3 = 3 \times 10^{-11}$. After about 10 vibrations, a molecule achieves thermal equilibrium. After about 20 vibrations, it escapes from its surrounding cage. After about 10 escape times, the molecule is in pressure equilibrium with the rest of the liquid. The fact that $10\tau_2 \approx \tau_3$ means that one diffusion time τ_3 (that is, one Stokes–Einstein diffusion time) is sufficient for both thermal and pressure equilibrium. Thus, Stokes–Einstein diffusion, while it appears to arise from a molecular kinetic theory and while it holds over a volume large enough to contain merely 20 or 30 molecules, is already close to a continuum theory. This is the justification for our using Navier–Stokes continuum theory, with appropriate "rolling" boundary conditions, for volumes perhaps as small as 3 molecules by 3 molecules by 3 molecules for continuum and then 2 more molecular diameters for rolling at the wall. That is, once a molecule is closely surrounded by neighbors, any movement which it makes outside that surrounding cage is already near continuum.

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